

Racecar Damping 4 – Applying State Space analysis to the Race car – PART 2

In our last instalment of race car damping we introduced and discussed the concept of using state space analysis in choosing damping for the race car. We discussed how to formulate the state space matrix, how it compared to the $\frac{1}{4}$ car approximation and we played with the $\frac{1}{4}$ car model to see how things compared.

In this instalment we will be taking what we learnt in part 1 and applying it to a bicycle model of a current spec F3 car. We're considering the case of the bicycle model of the F3 car so we can see the effects that aerodynamics has on the dynamics of the race car. It's also a very effective way of introducing inerters as well. The F3 car is a very interesting case because it marks the crossover where aerodynamics has a significant impact on vehicle performance. Consequently for our discussion it is perfect.

So that we are crystal clear let us outline the parameters of our F3 car. This is summarised in Table 1

Table – 1 – Parameters for an F3 car.

Parameter	Symbol	Value	Unit
Total mass	M_t	550	Kg
Total unsprung mass at the front	M_{tf}	20	Kg
Total unsprung mass at the rear	M_{tr}	30	Kg
Front spring rate	K_f	120000	N/m
Front Damper rate	C_f	12000/5000	N/m/s
Front tyre spring rate	K_{tf}	240000	N/m
Rear spring rate	K_r	100000	N/m
Rear Damper rate	C_r	12000/5000	N/m/s
Rear tyre spring rate	K_{tr}	260000	N/m
Wheelbase	Wb	2.7	M
Front weight distribution	Wdf	41%	
Distance from front axle to c.g	a	1.593	m
Distance from rear axle to c.g	b	1.107	m
Approximate C_{LA}		2.5	

To further clarify our discussion we are considering a twin spring car front and rear, and I have just quoted the rates at the corner. In reality in the bicycle model I'm doubling these numbers, but for clarity I've quoted at corner values. These numbers are also in wheel rates.

To save ourselves a bit of time I'm going to quote the linearised state space matrix. This is summarised in equation (1),

$$A = \begin{bmatrix} \frac{-(c_f + c_r)}{m_s} & \frac{-(k_f + k_r) + \frac{\delta F}{\delta z}}{m_s} & \frac{(a \cdot c_f - b \cdot c_r)}{m_s} & \frac{(a \cdot k_f - b \cdot k_r) + \frac{\delta F}{\delta \theta}}{m_s} & \frac{c_f}{m_s} & \frac{k_f}{m_s} & \frac{c_r}{m_s} & \frac{k_r}{m_s} \\ \frac{(a \cdot c_f - b \cdot c_r)}{I_y} & \frac{(a \cdot k_f - b \cdot k_r) + \frac{\delta M}{\delta z}}{I_y} & \frac{(-a^2 \cdot c_f - b^2 \cdot c_r)}{I_y} & \frac{(-a^2 \cdot k_f - b^2 \cdot k_r) + \frac{\delta M}{\delta \theta}}{I_y} & -a \cdot c_f & -a \cdot k_f & b \cdot c_r & b \cdot k_r \\ \frac{c_f}{m_g} & \frac{k_f}{m_g} & -a \cdot c_f & -a \cdot k_f & \frac{-(c_f + c_g)}{m_g} & \frac{-(k_f + k_g)}{m_g} & 0 & 0 \\ \frac{c_r}{m_r} & \frac{k_r}{m_r} & b \cdot c_r & b \cdot k_r & 0 & 0 & \frac{-(c_r + c_r)}{m_r} & \frac{-(k_r + k_r)}{m_r} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (1)$$

The reader should recognise equation (1) from my articles on how to analyse aeromaps that was published about 2 years ago. I leave the derivation of equation (1) to the reader (In reality it actually isn't that hard, it's just an extension of what we did in part 1).

However one area I do want to go over in a little detail is the derivation of the aero derivatives. The reader will recall I laid out the definition of the aero derivatives. I will recap these in equation (2),

$$\begin{aligned} \frac{\delta F_z}{\delta z} &= \frac{\partial}{\partial z} (C_L A) \cdot \frac{1}{2} \cdot \rho \cdot V^2 \\ \frac{\delta F_z}{\delta \theta} &= \frac{\partial}{\partial \theta} (C_L A) \cdot \frac{1}{2} \cdot \rho \cdot V^2 \\ \frac{\delta M}{\delta z} &= wb \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot \left[(wdf - ab) \cdot \frac{\delta}{\delta z} (C_L A) - \frac{\delta}{\delta z} (ab) \cdot C_L A \right] \\ \frac{\delta M}{\delta \theta} &= wb \cdot \frac{1}{2} \cdot \rho \cdot V^2 \cdot \left[(wdf - ab) \cdot \frac{\delta}{\delta \theta} (C_L A) - \frac{\delta}{\delta \theta} (ab) \cdot C_L A \right] \end{aligned} \quad (2)$$

At the time I mentioned you had to convert aeromaps from functions of front and rear ride height to functions of heave (z) and pitch angle (θ). Well in the process of researching this article I found a shortcut, and I'm kicking myself I didn't think of it 2 years ago.

The shortcut uses the chain rule of differentiation of several variables. The ace in the hole is the following,

$$\begin{aligned} rh_f &= rh_{f0} - (z - a \cdot \theta) \\ rh_r &= rh_{r0} - (z + b \cdot \theta) \\ a &= (1 - wdf) \cdot wb \\ b &= wdf \cdot wb \end{aligned} \quad (3)$$

So what we need to do is to construct partial derivatives of front and rear ride height as functions of heave and pitch angle. Using the chain rule of several variables it can be shown,

$$\begin{aligned}
\frac{\partial}{\partial z}(C_L A) &= -\left(\frac{\partial}{\partial r h_f}(C_L A) + \frac{\partial}{\partial r h_r}(C_L A) \right) \\
\frac{\partial}{\partial \theta}(C_L A) &= a \cdot \frac{\partial}{\partial r h_f}(C_L A) - b \cdot \frac{\partial}{\partial r h_r}(C_L A) \\
\frac{\partial}{\partial z}(ab) &= -\left(\frac{\partial}{\partial r h_f}(ab) + \frac{\partial}{\partial r h_r}(ab) \right) \\
\frac{\partial}{\partial \theta}(ab) &= a \cdot \frac{\partial}{\partial r h_f}(ab) - b \cdot \frac{\partial}{\partial r h_r}(ab)
\end{aligned} \tag{4}$$

The power of equation (4) is that we can use our existing aeromaps to readily construct our aero derivatives. That's pretty cool and I leave the derivation of equation (4) to the interested reader.

Armed with this knowledge I constructed state space matrices for the high speed damping case. This included the case without and with downforce. The results were very interesting. The non downforce results are shown in Table –2. For brevity I have rolled in the complex conjugate pairs,

Table – 2 Eigenvalues and eigenvectors for the non downforce case.

Mode	1	2	3	4	5	6
Eigenvalue	-503.7	-302.7	-55.3	-15.5+46i	-35.1	-11+26i
Eigenvectors						
Z	0.04	-0.08	-0.46	0.09-0.09i	-0.01	0.829
Z	0	0	0.01	0	0	-0.01-0.027i
θ	-0.07	-0.065	-0.09	-0.45+0.36i	-0.85	0.21-0.06i
θ	0	0	0	0.01+0.01i	0.02	0
z _{tf}	-0.99	0.1527	0.44	0.721	-0.38	0.14+0.18i
z _{tf}	0	0	-0.01	0	0.01	0
z _{tr}	0.06	0.98	0.76	-0.36 – 0.05i	0.35	0.32+0.33i
z _{tr}	0	0	-0.01	0	-0.01	-0.01i

As we can see there are six primary modes. Modes 1 and 2 act primarily on the velocities of the unsprung mass. These die off very quickly which is a good thing. Modes 3 and 4 have very similar frequencies (approximately 55.3 rad/s) and are combined modes that effect heave, pitch angle and the unsprung mass. Modes 5 and 6 dominate the time response and have frequencies approximately 35 rad/s, and act primarily on the sprung mass modes of heave and pitch. The first thing to point out is we have taken up where we left off with our ¼ car example from part 1. We saw very similar trends, but now we have added more detail.

The downforce results are presented in Table 3. The eigenvectors didn't change that much. However the eigenvalues did and the results are shown in Table 3

Table – 3 Eigenvalues for the downforce case @ 200 km/h.

Mode	1	2	3	4	5	6
Eigenvalue	-503.7	-302.7	-55.6	-16.9+44i	-32.4	-10.8+27.2i
Eigenvectors						
Z	-0.04	-0.08	-0.46	0.075-0.09i	0.03	0.827
Z	0	0	0.01	0	0	-0.01-0.027i
θ	0.07	-0.065	-0.09	-0.44+0.38i	-0.91	0.19-0.06i
θ	0	0	0	0.01+0.01i	0.03	0
z_{tf}	0.99	0.1527	0.44	0.712	-0.30	0.15+0.19i
z_{tf}	0	0	-0.01	0	0.01	0
z_{tr}	-0.06	0.98	0.76	-0.36 – 0.05i	0.285	0.33+0.32i
z_{tr}	0	0	-0.01	0	-0.01	-0.01i

Apart from the eigenvector for mode 1 the rest of them are very similar to the non downforce case. The reader will note that mode 1 is inversed to what it was in Table 2. However the magnitudes are almost identical. This means this mode is still acting primarily on the same way, which is primarily on the front unsprung mass, it's just acting in the opposite way. However this is all a bit academic as this mode dies off very fast.

In terms of the eigenvalues as we can see modes 1 – 3 didn't change that much. Mode 4 became a bit more damped, the frequency of mode 6 dropped a bit but mode 5 dropped by nearly 3 rad/s. What this indicates is that when ever a mode drops a natural frequency it indicates the system has been destabilised. This is exactly what we would expect a ride height sensitive aero car to do. Albeit in this case while this is something we have to deal with, it's certainly not something you'd be using sleep over.

However what Tables 2 and 3 tell us some general trends, that the more aero sensitive a car is the more you have to stabilise it. This situation is illustrated in this pole zero diagram as illustrated in Fig 1.

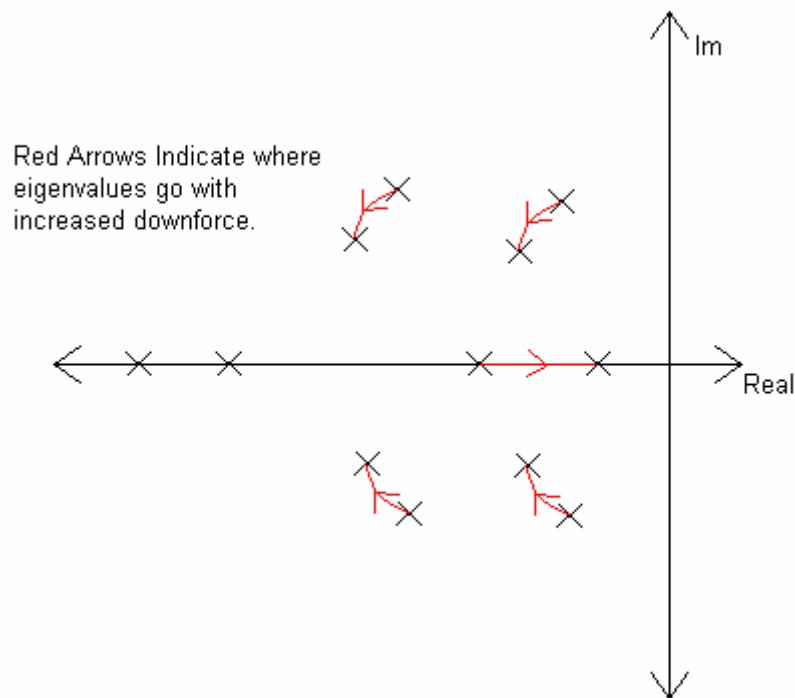


Fig-1 – Pole zero diagram as more downforce is applied.

As you can see the more downforce that is applied the less stable the more dominant mode is. This is the reason sports prototypes can't be run below a minimum ride height. They experience something called porpoising where the chassis rocks back and forth, and this is the extreme case illustrated in Fig 1.

At this point you might be thinking this is rather entertaining but so what? Remember the eigenvalues returned by the state space give you the natural frequency and damping ratio. For the reader's convenience let me remind you of the formulation,

$$\begin{aligned}\sigma \pm i \cdot \omega_d &= \zeta \omega_n \pm i \cdot \omega_n \cdot \sqrt{1 - \zeta^2} \\ \omega_n &= \sqrt{\sigma^2 + \omega_d^2} \\ \zeta &= \sqrt{1 - \left(\frac{\omega_d}{\omega_n}\right)^2}\end{aligned}\tag{5}$$

Remember what I said about the 1/4 car model. When it comes to aero and inerters it runs out of steam. Yet using the bicycle model, we have captured exactly what is going on and we can see what it is up to. Once we know the effects the aero is having on the car for a given setup, then we can change the spring and damper settings to deal with it. This is where equation (5) is such a powerful tool because we can nail this down with far greater clarity than what we could do with the 1/4 car approximation.

The next real powerful application of state space techniques is considering the effect of inerters. Some of you may recall an article I did about a year and a half ago where I used state space analysis to analyse and play the devils advocate on inerters. I have no desire to repeat word for word what I said in that article, but to help us with inerters I used the concept of an inertance matrix. The formulation was,

$$\dot{x} = (I_{inert})^{-1} A \cdot x \quad (6)$$

Here A was the typical state space equation as outlined in equation (1), and by choosing values of inerters we could construct the inertance matrix. Once we have equation (6) we can solve for the eigenvalues and eigenvectors and away we go. This is something we simply couldn't have done with the 1/4 car approximation.

With all of this fancy mathematical footwork I have presented the next question that has to be asked is where does this all fit in. To that end I have the following guide on what to do to specify dampers for the race car.

- Start by using the 1/4 car approximation technique we discussed in Part 2 of this series. Use the damper guide but just remember you can go a bit over. Remember the powerful thing about this is it is a quick hand calculation, which you can readily put into a setup sheet.
- Once you have your start points as specified by the 1/4 car approximation, then you can move on to state space analysis. If you need something quick and dirty use the quarter car. If you need to incorporate aero and inerters, use the bicycle model, and if you need to incorporate roll use the full beam pogo stick model. Remember don't be a hero, use Matlab, maple etc to do the eigenvalue, eigenvector stuff for you.
- Once you have refined your choices use simulation software like ChassisSim extensively to test the results. This is the major reason I used a transient simulation engine for ChassisSim so make good use of it.
- Finally always validate at the track.

Remember treat this as a guide and not as something in the same league as the Ten Commandments. Last time I looked I didn't have a direct line to any religious deity. However if you go through this process I guarantee it will get you pretty close.

In closing then the subject of race car damping is incredibly vast and complex and this article is certainly not the last word on the subject. I can tell you right now anybody who thinks they now it all on race car damping should be shot on site for their own safety. However I trust what I have presented here with the use of the 1/4 car approximation and the use of eigenvalues and eigenvectors provides a path through the forest. If that is all I have done then I have succeeded.